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A.E. BROUWER

ON THE NUMBER OF UNIQUE SUBGRAPHS OF A GRAPH

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Abstract

A question of Entringer and Erdös concerning the number of unique subgraphs of a graph is answered.

Entringer and Erdös [1] call a subgraph H of a graph G unique if H is not isomorphic to any other subgraph of G. If f(n) is the largest number of unique subgraphs a graph on n vertices can have, they prove

$$f(n) > 2^{\frac{1}{2}n^2 - cn^{3/2}} \qquad \text{for } c > 3/\sqrt{2} \text{ and n sufficiently large}$$

It will be proved below that

$${}^2\log f(n) = \tfrac{1}{2}n^2 - n \cdot {}^2\log n + O(n).$$

Since the number of nonisomorphic graphs on n vertices is

$$\frac{2^{\binom{n}{2}}}{n!}\left(1 + \frac{n^2-n}{2^n-1} + O\left(\frac{n^3}{2^{3n/2}} \right)\right)$$

(see e.g. [2], p.196), we have

$${}^2\log f(n) \le \tfrac{1}{2}n^2 - n.\cdot {}^2\log n + O(n).$$

On the other hand, given n we construct a graph G_n on n vertices with $2^{\frac{1}{2}n^2-n\cdot {}^2\log n+O(n)}$ unique subgraphs as follows:
Let $m = \lceil {}^2\log n \rceil$ and $N = n - m - 2$. Then $N \le 2^m - m - 1$.
Let $G_n = A \cup B \cup C$ where

$A = K_N$, the complete graph on N points,
B is a rigid tree with m vertices (such a tree exists for each $m \ge 7$),
$C = K_2$, a single edge connecting points c_0 and c_1 , connected as follows:

G_n contains all edges (c_1,b) for $b \in B$ and no other edges between C and $A \cup B$; furthermore, if we view A as a set of subsets of B each containing at least two points (which is possible since $N \le 2^m-m-1$), then G_n contains the edge (a,b) where $a \in A$ and $b \in B$ if and only if $b \in a$.

Now define the subgraph H_n of G_n as follows:

$H_n = A' \cup B \cup C$ where A' is the vertex graph on N vertices (that is, A' is totally disconnected) and A', B, C are interconnected like A, B, C in G_n. That is, H_n contains the same n points as G_n but has $\binom{N}{2}$ edges less.

If H is a subgraph of G_n such that $H_n \subset H \subset G_n$ then H is unique: First, c_0 is the only point of H with degree one, and therefore if we imbed H in G_n the point c_0 of H must go to the point c_0 of G_n. Next it follows that c_1 must go to c_1 and therefore that $B \subset H$ must map onto $B \subset G$. Since B is rigid the imbedding restricted to B must be the identity on B. Finally, since each point of A is coded by a subset of B, A too cannot be imbedded in any other way. Therefore H is unique.

The number of subgraphs H between H_n and G_n being $2^{\binom{N}{2}} =$
$= 2^{\frac{1}{2}n^2-n \cdot {}^2\log n+O(n)}$, this proves our assertion.

References

[1] R.C. Entringer and Paul Erdös, *On the number of unique subgraphs of a graph*, JCT (B) 13, 112-115 (1972).

[2] Frank Harary & Edgar Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.